1 Exterior Differential System

Let M^n be a smooth $\mathcal{C}^n$-manifold and $\phi_1, ..., \phi_s$ be differential forms on M^n. The exterior differential system is a system of equations

$$\phi_i = 0, \quad i = 0, ..., s.$$

The goal is to find a submanifold of M^n on which $\phi_i = 0$. Especially, if all ϕ_i's are 1-forms, it is called a Pfaffian system.

Definition 1.1. An integral manifold is an immersion $f : N \to M$ such that

$$f^*\phi_i = 0, \quad i = 0, ..., s.$$

If $f : N \to M$ is an integral manifold, then $f^*(d\phi_i) = d(f^*\phi_i) = 0$ and $f^*(\psi \wedge \phi_i) = f^*\psi \wedge f^*\phi_i = 0$ for any form ψ on M. Thus we are really working with the differential ideal generated by $\{\phi_1, \ldots, \phi_s\}$.

Example 1.2 (Pfaff problem). In $\mathbb{R}^n$, let

$$x = (x^1, \ldots, x^n)$$

be the coordinates of $\mathbb{R}^n$ and

$$\omega = a_1(x)dx^1 + \cdots + a_n(x)dx^n.$$

Clearly, the equation $\omega = 0$ has a solution since we have an integral curve of a vector field which is orthogonal to $(a_1, \cdots, a_n)$ by the existence theorem of ordinary differential equations. We want to find a k $(< n)$ dimensional integral manifold $f : \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^k$. Put

$$f = (f^1, \ldots, f^n)$$

and let

$$y = (y^1, \ldots, y^k)$$

be the coordinates of $\mathbb{R}^k$. Then

$$\begin{aligned} f^*\omega &= a_1(f(y))df^1 + \cdots + a_n(f(y))df^n \\ &= \sum_{\lambda=1}^{k}\left(a_1(f(y))\frac{\partial f^1}{\partial y^\lambda} + \cdots + a_n(f(y))\frac{\partial f^n}{\partial y^\lambda}\right)dy^\lambda \\ &= 0. \end{aligned}$$

Therefore

$$a_1(f(y))\frac{\partial f^1}{\partial y^\lambda} + \cdots + a_n(f(y))\frac{\partial f^n}{\partial y^\lambda} = 0, \quad \lambda = 1, \ldots, k.$$

This is an underdetermined system of PDE with n unknowns and k equations. The Pfaff problem is finding an integral manifold of maximal dimension.

We use the following notations :

(i) $\Omega^0(M) = C^\infty(M)$: the 0-forms,

(ii) $\Omega^p(M)$: the set of smooth p-forms on M for $p = 1, \ldots, n$,

(iii) $\Omega^*(M) = \bigoplus_{p=0}^{n} \Omega^p(M)$: graded module over $C^\infty(M)$.

$\{\Omega^*(M), \wedge, d\}$ is called **exterior algebra of differential forms**.

Definition 1.3. Exterior differntial system(EDS) is a pair $(M, \mathfrak{I})$ where M is a smooth manifold and $\mathfrak{I} \subset \Omega^*(M)$ is an ideal in the graded ring $\Omega^*(M)$ of differential forms on M that is closed under exterior differentiation, that is, $d\phi \in \mathfrak{I}$ for any $\phi \in \mathfrak{I}$.

Definition 1.4. A subalgebra $\mathfrak{I} \subset \Omega^*(M)$ is called an **(algebraic) ideal** if the following are satisfied.

(i) If $\phi \in \mathfrak{I}$, then $\psi \wedge \phi \in \mathfrak{I}$ for any $\psi \in \Omega^*(M)$.

(ii) If $\phi \in \mathfrak{I}$, each homogeneous component of ϕ is in $\mathfrak{I}$.

Definition 1.5. A subalgebra $\mathfrak{I} \subset \Omega^*(M)$ is called a **differential ideal** if

(i) $\mathfrak{I}$ is an algebraic ideal,

(ii) $d\mathfrak{I} \subset \mathfrak{I}$, that is, if $\phi \in \mathfrak{I}$, then $d\phi \in \mathfrak{I}$.

Thus definition 1.3 implies that an EDS is a pair $(M, \mathfrak{I})$, where M is a smooth manifold and $\mathfrak{I} \subset \Omega^*(M)$ is a differential ideal.

Let $\mathfrak{I} \subset \Omega^*(M)$ be an algebraic ideal. Then $\mathfrak{I} = \bigoplus_{q=0}^{n} \mathfrak{I}^q$, where $\mathfrak{I}^q = \mathfrak{I} \bigcap \Omega^q(M)$. Hence $\mathfrak{I}$ itself is a graded algebra.

In most cases, generators are given: $\phi_1, \ldots, \phi_s \in \Omega^*(M)$. The algebraic ideal $< \phi_1, \ldots, \phi_s >_{alg}$ generated by $\{\phi_1, \ldots, \phi_s\}$ is the set of differential forms $\phi = \gamma^1 \wedge \phi^1 + \cdots + \gamma^s \wedge \phi^s, \quad \gamma^j \in \Omega^*(M)$ and the differential ideal $< \phi_1, \ldots, \phi_s >$ generated by $\{\phi_1, \ldots, \phi_s\}$ is the set of differential forms $\phi = \gamma^1 \wedge \phi^1 + \cdots + \gamma^s \wedge \phi^s + \beta^1 \wedge d\phi^1 + \cdots + \beta^s \wedge d\phi^s, \quad \gamma^j, \beta^k \in \Omega^*(M)$, that is, the algebraic ideal generated by ϕ's and $d\phi$'s.

The fundamental problem in EDS is to study integral manifolds of differential ideals.

Let Ω be a decomposable p-form,

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p, \qquad \omega^j : \text{1-form}$$

and $\mathfrak{I}$ a differential ideal. Then the pair $(\mathfrak{I}, \Omega)$ is called an **EDS with independence condition** Ω. The integral manifold of $(\mathfrak{I}, \Omega)$ is an integral manifold of $\mathfrak{I}$ such that $f^*\Omega \neq 0$. We use this system when we wish to keep some variables independent.

Remark. Every PDE(ODE) system can be written as an EDS with independence condition. A couple of examples are shown below. The independent variables of PDE make the independence condition of the EDS.

Example 1.6. Consider the PDE of order 2

$$y'' = F(x, y, y').$$

Then we obtain $dy = y'dx$ and $dy' = y''dx = F(x, y, y')dx$. Thus, on $M = \{(x, y, y')\} = \mathbb{R}^3$, the above PDE gives an EDS of 1-forms

$$\begin{cases} dy - y'dx, \\ dy' - Fdx, \end{cases}$$

with independence condition $dx \neq 0$.

Example 1.7. Consider the PDE of order 2

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Let M be the second jet space $J^2(\mathbb{R}^2, \mathbb{R})$ such that

$$M = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\} = \mathbb{R}^8 = \{(x, y, u, p, q, r, s, t)\}.$$

We obtain the EDS made by

0-form $\quad F(x, y, u, p, q, r, s, t) = 0,$

$$\text{1-forms} \begin{cases} du &= u_x dx + u_y dy &= pdx + qdy &\rightarrow du - pdx - qdy &= 0, \\ du_x &= u_{xx}dx + u_{xy}dy &= rdx + sdy &\rightarrow dp - rdx - sdy &= 0, \\ du_y &= u_{xy}dx + u_{yy}dy &= sdx + tdy &\rightarrow dq - sdx - tdy &= 0 \end{cases}$$

with independence condition $dx \wedge dy \neq 0$.

Let $\alpha^1, \ldots, \alpha^{n-r}$ be the given 1-forms on M^n which are independent and $\mathfrak{I}$ the ideal generated by $\alpha^1, \ldots, \alpha^{n-r}$. $\mathfrak{I}$ is said to be **closed** if it satisfies the following condition:

$$\left.\begin{aligned} & d\mathfrak{I} \subset \mathfrak{I} \\ \Leftrightarrow\quad & d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{n-r} \\ \Leftrightarrow\quad & d\alpha^i = \phi_1 \wedge \alpha^i + \cdots + \phi_{n-r} \wedge \alpha^{n-r} \end{aligned}\right\} \qquad (1)$$

A Pfaffian system $\alpha^i = 0$, $i = 1, \ldots, n-r$ is called completely integrable if the condition (1) holds.

Theorem 1.8 (Frobenius, [1]). *Let $\mathfrak{I}$ be a differential ideal generated by 1-forms $\alpha^1, \ldots, \alpha^{n-r}$ so that the condition (1) is satisfied. Then, in a sufficiently small neighborhood, there exists a coordinate system $y^1, \ldots, y^n$ such that $\mathfrak{I}$ is generated by $dy^{r+1}, \ldots, dy^n$.*

Example 1.9. In $\mathbb{R}^3$, let $\omega = Rdx + Sdy + Tdz$. Then $d\omega \equiv 0 \mod \omega$ if and only if there exists a function μ such that $\mu\omega$ is exact.

2 Jet Bundle(Jet Space)

Let N and M be manifolds of dimensions k and n, respectively. For each $r = 0, 1, 2, \ldots$, the r-th **jet space(jet bundle)** is roughly the set of all partial derivatives up to order r of maps $f : N \to M$.

Definition 2.1. The maps f, $g : N \to M$ are said to have the **same r-th jet** at p if partial derivatives of f and g up to order r are equal. Then the relation is an equivalence relation and the equivalence class with the representative $f : N \to M$ is denoted by $j_p^r(f)$. Let $J_{p,q}^r$ denote the set of all r-jets of mappings from N into M with source p and target q. Then define the set

$$J^r(N, M) = \bigcup_{p \in N, q \in M} J_{p,q}^r(N, M).$$

$J_{p,q}^r$ is the doubly fibred manifold with the natural projections α and β, where $\alpha : J^r(N, M) \to N$ and $\beta : J^r(N, M) \to M$ defined by $\alpha(j_p^r(f)) = p$ and $\beta(j_p^r(f)) = f(p)$.

Let (U, x) and (V, z) be the coordinate charts of N and M, respectively. Then $\alpha^{-1}(U) \cap \beta^{-1}(V)$ is a coordinate neighborhood of $J^r(N, M)$. We may define a coordinate system by

$$h(j_p^r(f)) = \big(x^i(p), z^j(f(p)), D_x^\alpha(z \circ f)(p)\big),$$
$$1 \leq i \leq k, 1 \leq j \leq n, 1 \leq |\alpha| \leq r.$$

The chain rule guarantees that a differentiable change of local coordinates in N and M will induce a differentiable change of local coordinates in $J^r(N, M)$.

The r-th graph $j^r(f) : N \to J^r(N, M)$ of a map f is defined by $j^r(f)(p) = j_p^r(f)$. On $J^r(N, M)$, we write the natural coordinates as

$$x^i(p), z^\alpha(f(p)), p_i^\alpha, p_{i_1,i_2}^\alpha, \ldots, p_{i_1,\cdots,i_r}^\alpha, \quad 1 \leq i, i_1, \ldots, i_r \leq k,\ 1 \leq \alpha \leq n.$$

Then the Pfaffian system

$$\begin{cases} dz^{\alpha} - p^{\alpha}_{i} dx^{i}, \\ p^{\alpha}_{i_1} - p^{\alpha}_{i_1,i_2} dx^{i_2}, \\ \quad \vdots \\ p^{\alpha}_{i_1,\ldots,i_{r-1}} - p^{\alpha}_{i_1,\ldots,i_{r-1},i_r} dx^{i_r} \end{cases}$$

is called the **contact system** $\Omega^r(N, M)$.

References

[1] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems* (1991), Springer-Verlag, New York, Berlin, Heidelberg.

[2] R. Bryant, P. Griffiths and D. Yang, *Characteristics and existence of isometric embeddings*, Duke Math. J. **50** (1983), 893-994.